

RATIONAL AND REAL POSITIVE SEMIDEFINITE RANK CAN BE DIFFERENT

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ABSTRACT. Given a nonnegative matrix M with rational entries, we consider two quantities: the usual positive semidefinite (psd) rank, where the matrix is factored through the cone of real symmetric psd matrices, and the rational-restricted psd rank, where the matrix factors are required to be rational symmetric psd matrices. It is clear that the rational-restricted psd rank is always an upper bound to the usual psd rank. We show that this inequality may be strict by exhibiting a matrix with psd rank four whose rational-restricted psd rank is strictly greater than four.

The *positive semidefinite (psd) rank* of a matrix was introduced in [GPT13] (see also [FMP⁺12]). In this note, we answer a basic structural question about the psd rank: if a nonnegative matrix M has only rational entries, can the psd rank of M always be achieved by a factorization using only rational matrices? We answer this question negatively by providing an example of a rational matrix with psd rank four such that every psd factorization of size four uses irrational numbers. Note that the analogous question for the *nonnegative rank* of a matrix was posed by Cohen and Rothblum in [CR93]. It was shown in [CR93] that all rational matrices with nonnegative rank two admit a rational nonnegative factorization, but the question for general nonnegative matrices remains open.

Let \mathcal{S}_+^k denote the cone of real symmetric $k \times k$ psd matrices. The psd rank of a nonnegative matrix is defined as follows:

Definition 1. Given a nonnegative matrix $M \in \mathbb{R}_+^{p \times q}$, a *psd factorization* of M of size k is a collection of psd matrices $A_1, \dots, A_p \in \mathcal{S}_+^k$ and $B_1, \dots, B_q \in \mathcal{S}_+^k$ such that $M_{ij} = \langle A_i, B_j \rangle$ for all $i = 1, \dots, p$ and $j = 1, \dots, q$, where the inner product is the standard trace inner product on symmetric matrices. The *psd rank* of M , denoted $\text{rank}_{\text{psd}} M$, is the smallest integer k for which M admits a psd factorization of size k .

The proof of our example will require a lemma about rational psd matrices of rank one. Any rank one psd matrix has the form $\mathbf{v}\mathbf{v}^T$ for some vector \mathbf{v} . Let ϕ denote the map taking the vector \mathbf{v} to the psd matrix $\mathbf{v}\mathbf{v}^T$. Then we have the following.

Lemma 2. *If the matrix $\phi(\mathbf{v})$ is composed of only rational entries, then \mathbf{v} has the form $\alpha \mathbf{q}$ where α is a real scalar and \mathbf{q} is a rational vector.*

Proof. Suppose that \mathbf{v} is a nonzero vector (else the conclusion is immediate). Then, without loss of generality, we may assume that the first coordinate v_1 is nonzero. Since v_1^2 is an entry in the matrix $\phi(\mathbf{v})$, it must be rational. Hence, the matrix $\frac{1}{v_1^2} \phi(\mathbf{v})$ is also rational. By looking at the first row of this matrix, we see that the vector $\left(1, \frac{v_2}{v_1}, \frac{v_3}{v_1}, \dots, \frac{v_r}{v_1}\right)$ is rational. Now we just scale this rational vector by v_1 to finish the proof. \square

Our candidate matrix M is the 8×6 matrix shown in Figure 1. Readers who are familiar with slack matrices may be interested to know that M arises as a slack matrix of the polytope with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 2, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 2, 1)$. Readers who are not familiar with slack matrices need not worry, as we will refrain from using any results about slack matrices in the proofs.

During our analysis of this example, we will require a few results about psd rank found in the literature. We summarize these results in the following proposition.

Proposition 3.

- (1) [GPT13, Prop. 5] *If \sqrt{A} is an entry-wise square root of A , then $\text{rank}_{\text{psd}} A \leq \text{rank } \sqrt{A}$.*
- (2) [LT12, Cor. 4.8], [GRT13, Prop. 2.6] *If A contains a $k \times k$ triangular submatrix T , then $\text{rank}_{\text{psd}} A \geq k$. Furthermore, in a psd factorization of A of size k , the factor corresponding to the row (or column) of T with $k - 1$ zeros must have rank one.*

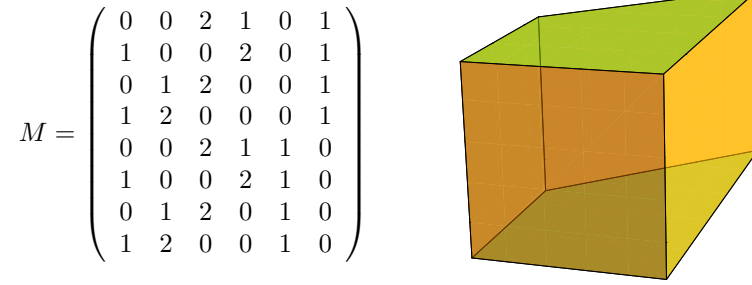


Figure 1. Our example matrix is a slack matrix of a three dimensional polytope.

Now we begin our analysis of the matrix M .

Lemma 4. *We have that $\text{rank}_{\text{psd}} M = 4$ and any psd factorization of M of size four uses only rank one factors.*

Proof. One can verify that the all-nonnegative entry-wise square root of M has usual rank four. Thus Proposition 3 says that $\text{rank}_{\text{psd}} M \leq 4$. Consider the submatrix of M indexed by rows 1, 5, 7, and 8 and columns 1, 2, 5, and 6. This submatrix is triangular so Proposition 3 tells us two things: First, we have that $\text{rank}_{\text{psd}} M \geq 4$ and, hence, $\text{rank}_{\text{psd}} M = 4$. Second, the factors corresponding to the first row and first column in a psd factorization of M of size four must always be rank one. It is easy to verify by inspection that for every row and column of M we can find a 4×4 triangular submatrix such that the row or column in question has three zeros in that submatrix. Thus, repeatedly applying the proposition completes the proof. \square

Remark 5. Note that Lemma 4 is actually a consequence of [GRT13, Prop. 3.2] since our polytope has minimal psd rank (equal to the ambient dimension plus one) and thus any psd factorization must consist entirely of rank-one factors.

The next proposition shows that no rational psd factorization of M can have size four.

Proposition 6. *We have that $\text{rank}_{\text{psd}} M = 4$, but there does not exist a psd factorization of size four using only rational matrices.*

Proof. Suppose, by way of contradiction, that $(A_1, \dots, A_8, B_1, \dots, B_6)$ is a psd factorization of M of size four that uses only rational matrices. By Lemma 4, each matrix must be rank one. Thus, there exist vectors $\mathbf{a}_1, \dots, \mathbf{a}_8$ and $\mathbf{b}_1, \dots, \mathbf{b}_6$ such that $A_i = \phi(\mathbf{a}_i)$ and $B_j = \phi(\mathbf{b}_j)$. Furthermore, by the properties of the trace, we must have that $M_{ij} = \langle A_i, B_j \rangle = \langle \mathbf{a}_i, \mathbf{b}_j \rangle^2$. Thus, the matrix whose (i, j) th entry is given by $\langle \mathbf{a}_i, \mathbf{b}_j \rangle$ is an entry-wise square root of M , which we denote by S . By looking at the submatrix generated by the first two rows and the fourth and sixth columns, we see that S contains a submatrix \tilde{S} of the form

$$\begin{pmatrix} \pm 1 & \pm 1 \\ \pm \sqrt{2} & \pm 1 \end{pmatrix}$$

where there is ambiguity on the sign of each entry.

Now by Lemma 2, each \mathbf{a}_i and \mathbf{b}_j must be a rational vector scaled by a nonzero real number. Hence, there must exist nonzero real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that the matrix resulting from the product

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \cdot \tilde{S} \cdot \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} = \begin{pmatrix} \pm \alpha_1 \beta_1 & \pm \alpha_1 \beta_2 \\ \pm \sqrt{2} \alpha_2 \beta_1 & \pm \alpha_2 \beta_2 \end{pmatrix}$$

is rational. It is easy to see that if $\alpha_1 \beta_1$, $\alpha_1 \beta_2$, and $\alpha_2 \beta_2$ are rational, then $\alpha_2 \beta_1$ must also be rational, which results in a contradiction. \square

REFERENCES

- [CR93] J.E. Cohen and U.G. Rothblum. Nonnegative ranks, decompositions, and factorizations of nonnegative matrices. *Linear Algebra and its Applications*, 190:149–168, 1993. 1

- [FMP⁺12] Samuel Fiorini, Serge Massar, Sebastian Pokutta, Hans Raj Tiwary, and Ronald de Wolf. Linear vs. semidefinite extended formulations: Exponential separation and strong lower bounds. In *Proceedings of the Forty-fourth Annual ACM Symposium on Theory of Computing*, STOC '12, pages 95–106. ACM, 2012. [1](#)
- [GPT13] João Gouveia, Pablo A. Parrilo, and Rekha R. Thomas. Lifts of convex sets and cone factorizations. *Mathematics of Operations Research*, 38(2):248–264, 2013. [1](#)
- [GRT13] João Gouveia, Richard Z. Robinson, and Rekha R. Thomas. Polytopes of minimum positive semidefinite rank. *Discrete & Computational Geometry*, 50(3):679–699, 2013. [1](#), [2](#)
- [LT12] T. Lee and D.O. Theis. Support-based lower bounds for the positive semidefinite rank of a nonnegative matrix, 2012. [1](#)

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